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13. ABSTRACT (Maximum 200 words) We have attacked the problem of designing efficient time-frequency computational tools by: (a) Developing selection procedures which shape an analyzing signal from a priori and precomputed front-end computations on input data based on Zak tranform and ambiquity function. (b) Implementing and comparing code for computing Gabor coefficients based on methods found in [3, 11]. This code uses fast FFT algorithms developed under DARPA contract F49620-89-C-0020. We have determined that the algorithm based on the deconvolution formula in [4] produces the fastest code and have applied this code using the one-sided exponential window to transient signal detection. (c) Developed a new algorithm for computing classical Gabor coefficients based on the concept of a generalized biorthogonal. This algorithm "delays" the effects of zero theorem and provides for numerically stable computation of Gabor coefficients locally around known Gabor coefficients. (d) Developed the proper form of finite discrete Gabor transform by periodizing and sampling and presented the results in [10, 32]. We have applied these results to Gabor representational schemes for submicron filtering, image reconstruction and image transfer for application to submicron lithography and to designing and constructing optical devices to implement time-frequency representations and to carry out processing on such representations including transient signal detection.

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1 Introduction

This is the final technical report on a project which proposed to study the applicability of several time-frequency representations including the ambiguity function [7, 8, 9, 25, 35], Wigner-Ville distribution [33] the Gabor transform [19] and the wavelet transform [17, 18] to problems in nonlinear filtering, multichannel signal analysis, and detection and feature extraction especially of transient signals corrupted by non-Gaussian noise. The main contributions of this research are

- The development of adaptive methods for selecting windows or analyzing signals targeted to a given application which lead to compact representations and fast, numerically stable algorithms for analysis, synthesis and processing.
- A study comparing the applicability of several time-frequency representations to problems in signal detection and feature extraction which includes the effects of such representations on processing.
- The design and comparison of code implementing time-frequency signal representation and synthesis including the discovery of new algorithms with significantly different behavior from those in standard use.
- The application and construction of optical devices for implementing and processing time-frequency signal expansions.

Many important application areas including radar and sonar, seismology and medical imaging require tools which encode and process joint localized time and frequency information. Classical Fourier methods average information taken over entire signal duration and as such do not directly display this localized information. Time-frequency representations offer the potential of analyzing and processing signal information in a space where this localized time-frequency information is more readily available and can be exploited in nonlinear filtering, pattern recognition, classification and detection.

Due to the untimely death of professor George Eichmann of CCNY, professor Yao Li and professor Michael Conner of CCNY directed most of the research into applications with direct contract support. Professor Izdor Gertner and professor Joshua Zeevi of Rutgers University played collaborative roles in formulating specific applications some of which they independently tested. The mathematical theory as well as overall direction was the responsibility of professor Richard Tolimieri of CCNY who received additional support for this effort from DARPA contract #F49620 89 C0020. Some aspects of the mathematical research were carried out jointly with Richard Orr of Atlantic Aerospace Corporation.

2 Technical Problem

Classical Fourier methods are based on the stationarity assumption on the signals and Gaussian assumption on noise. Processing techniques are, therefore limited in most cases to the application of time-invariant systems to signals. The space of the signal (the time domain) and the space of its Fourier transform (the frequency domain) contain equivalent information. Filtering can be implemented in either domain by convolving the signal with a mask in the time domain or by multiplying the Fourier transform of the signal by a filtering function. In either case, the effect is to suppress, pass through or enhance certain signal frequencies uniformly across the signal. Since the Fourier transform averages information taken over all time, all points are equally affected.

The stationarity condition is not valid in many applications and does not faithfully represent the underlying natural process. Traditional techniques based on Fourier analysis do not exhibit the necessary resolution for dealing with many important signal processing tasks including those required in sonar processing where propagation and multi-path conditions must be taken into account. Time-frequency representational methods provide signal representations containing joint localized time-frequency information offering high resolution and good discrimination capabilities when non-stationarities and nonlinear effects distort the signal. Processing in the space of this representation can take place which modifies, selects, and distinguishes certain frequencies as before but now the effects of this processing vary over time providing powerful tools for dealing with nonstationary phenomena.

A major distinction between Fourier methods and time-frequency methods is that in the latter we have the freedom to choose a localizing window or analyzing signal relative to which signal information is referenced. The choice of the analyzing signal can affect

- The existence, uniqueness, and compactness of the representation.
- The speed and stability of computations.

The feasibility of time-frequency methods depends to a large extent on the existence of fast, numerically stable algorithms for synthesis and analysis of signals and for processing in the time-frequency domain.

As in the classical Fourier analysis of signals, performance is greatly affected by strategies which exploit known signal information. For example signal information as to time duration and bandwidth are essential for selecting a sampling rate which adequately represents the signal. The matched filter is the optimum signal processing strategy for signal detection and feature extraction in noisy environments when the signal is exactly known. Preprocessing strategies for time-frequency representations must give information as to the form and shaping of the window to be used in the analysis. We have based such strategies on the Zak transform and the ambiguity function.

3 Technical Results

We have attacked the problem of designing efficient time-frequency computational tools by

- Developing selection procedures which shape an analyzing signal from a priori and precomputed front-end computations on input data based on Zak transform and ambiguity function.

- Implementing and comparing code for computing Gabor coefficients based on methods found in [3, 11]. This code uses fast FFT algorithms developed under DARPA contract #F49620 89 C0020. We have determined that the algorithm based on the deconvolution formula in [4] produces the fastest code and have applied this code using the one-sided exponential window to transient signal detection.

- Developed a new algorithm for computing classical Gabor coefficients based on the concept of a generalized biorthogonal. This algorithm "delays" the effects of zero theorem and provides for numerically stable computation of Gabor coefficients locally around known Gabor coefficients.

- Developed the proper form of finite discrete Gabor transform by periodizing and sampling and presented the results in [10, 32].

We have applied these results to

- Gabor representational schemes for submicron filtering, image reconstruction and image transfer for application to submicron lithography

- Designing and constructing optical devices to implement time-frequency representations and to carry out processing on such representations including transient signal detection.

3.1 Zak Transform Selection Procedures

The study of selection procedures in the case of the Gabor transform has been carried out in some detail. Any Gabor scheme which seeks to apply to all square integrable signals will be limited by the Heisenberg uncertainty principle. The zero theorem [6] and more generally Balian's theorem are concrete ramifications of the uncertainty principle which directly obstruct general representation schemes, either by restricting the class of analyzing signals or by requiring oversampling *i. e.* lattices whose fundamental region area is less than one. The effort in this research was directed to the design of adaptive procedures which used front-end computation of the Zak transform of incoming data as tools for choosing optimal windows. The Zak transform [36] has played a major role in this research. It has played a crucial role in much of the theory and application of time-frequency representations including the design of fast algorithms for determining Gabor coefficients [3, 4, 11, 12]. The numerical stability of these algorithms is limited by the Zero-theorem [5]. These algorithms usually involve division by the Zak transform of the analyzing signal which if sufficiently smooth must have a zero. Results in [1, 30] suggests methods for classifying and reducing the negative consequences of the Zero-theorem.

The Zero-theorem is intimately tied to the Heisenberg uncertainty principle. One of the first proofs [5] rests on the non-abelian group structure of the Heisenberg group. In [21], we review the function theory on the Heisenberg group, more accurately on certain compact nil-manifolds coming from the Heisenberg group called Heisenberg manifolds and apply this theory to the problem of designing adaptive robust Gabor procedures. Many of these results are special to the Gaussian and make use of detailed knowledge of the Taylor series expansion of the Zak transform of the Gaussian at its unique zero in the unit square. Starting with the Gaussian, we can define, in a natural manner, a family of Gaussian-like analyzing signals parameterized by the number and the position of Zak transform zeroes. In fact, several families can be constructed offering some variation in the local behavior of the Zak transform at vanishing points and global envelope characteristics.

These results provide powerful tools for matching analyzing signal to application signal. In [5, 30], this led to a detailed description of the oversampling needed to remove the numerical instability of expanding arbitrary square integrable signals as integer time-frequency translates of the Gaussian. A more exhaustive analysis [30] established sufficient conditions on signals for the existence of good Gabor expansions.

Recent work has extended these results to signals other than the Gaussian such as Hermite functions [1], one-sided exponentials and Legendre functions. As with the Gaussian a family of analyzing signals can be associated to each signal of a type. The analyzing signals in a family can as before be parameterized by the number and the position of Zak transform zeroes however as we move through the families we have a variety of local behavior at the Zak transform zeroes and global Zak transform envelope characteristics.

3.2 Ambiguity Function Selection Procedures

In the early 1960s, research summarized in [16] into the Radar ambiguity function yielded a rather a complete analysis and characterization of the auto- and cross-ambiguity function. In [24], this research was coupled with the Poisson summation formula to derive fundamental formulas which lie at the heart of the theory of Gabor expansions. These formulas relate the energy of the discrete cross-ambiguity function of two signals over a lattice with the inner product of the discrete auto-ambiguity function of the signals over a complimentary lattice. The lattice in the former defines the lattice of time-frequency translates of the analyzing signal in the corresponding Gabor expansion. Frame conditions on analyzing signals and lattices imply a good representation theory for all square integrable functions. We will extend the role played by these fundamental formulas and use them to provide tools for shaping good analyzing signals and determining good time-frequency translation lattices for a given application. Although at present these results are more speculative as compared to Zak transform results, we expect that significant contributions to both theory and application of Gabor expansions will be achieved.

3.3 New Algorithms

In [11, 12, 13, 14], the Zak transform was used to construct a biorthogonal of an analyzing signal. This led to a computational procedure for computing Gabor coefficients by taking inner products relative to a biorthogonal but the Zero theorem once again created problems. For analyzing signals having continuous Zak transforms the biorthogonal need not be square- integrable. In the case of the Gaussian, we have introduced the concept of a generalized biorthogonal [32] which depending on certain moment conditions can be taken as smooth as desired but at a cost. The smoother the generalized biorthogonal the more complex and numerically unstable the algorithm computing Gabor coefficients. Direct inner product computations are no longer sufficient and certain difference equations must solved. However Gabor coefficient computation is localized in the sense that if a Gabor coefficient is known then in some region in the lattice around this Gabor coefficient, other Gabor coefficients can be efficiently computed. The extent to which this statement can be made precise and the role of such an algorithm in applications is currently under study.

3.4 Optical Processing

One aspect of this program was trying to identify the role of optics in the implementation and processing of wavelets and other promising decompositions. Through our two years of theoretical and experimental investigations, it has become quite clear that for many signal processing and radar applications, optics may play an important role to speed up the entire process. Our findings have been published in five refereed journal articles with one additional article in the process of publication. A brief summary of the findings is give below.

Gabor expansion of one dimensional short signal sequences has been generated optically for the first time. Through the use of the biorthogonal signal of the selected Gabor window, the expansion of an arbitrary signal upon the window was performed optically using a modified optical ambiguity function generator. In our experimental effort, the single-sided exponential function was selected as the Gabor window for the detection of transients. An acousto-optic processor to implement the approach was proposed and conceptually demonstrated.

Wavelet expansion which can overcome many disadvantages inherent to the Gabor expansion was also studied for its optical implementation for the first time. It was found that optics is quite suitable to generate and display both the direct and the inverse wavelet transforms in parallel. Unlike the digital computer-based implementation which prefers the course-to-fine serial generation procedure, using optics, the one-dimensional to two-dimensional coordinate expansion could be performed in parallel and on-the-fly. An optical processor to process one dimensional short data sequence was first built and tested. The processor contains one DC band and four other higher frequency channels. The decomposition of a spatial chirp signal into these bands and the reconstruction of the chirp using the wavelet decomposed signals were successfully performed for the first time.

A further step was taken toward identifying the suitability of using optics for the multichannel signal analysis. Both the Gabor and the wavelet transforms were studied in terms of their complexity of optical implementations in general. Both one and two dimensional signal cases were assumed. It was found that the role of optics in the Gabor case was to process the ambiguity function necessary for the implementation. While for the signal recovery, although it is not impossible, it is very difficult to use the optical method. Also, due to the use of coherent processing, it is the square of, not the Gabor coefficients themselves, which is finally generated. Thus, the optical methods are more suitable for the detection of certain signal signatures based on the selected window function than using it as a mathematical expansion tool. On the other hand, optics finds itself promising for generating both direct and inverse wavelet transforms. This is the case for processing both the one and two dimensional signals. A detail comparison of the space-bandwidth product related, filter dynamic range limited, as well as the misalignment caused performance degradations for both applications was performed.

To extend the method of processing the Gabor and wavelet transforms of one dimensional short signal sequence to a more practical situation where the one dimensional long data sequence are encountered, a novel optical processor concept which scans the input long data sequence into a two dimensional format was proposed and numerically simulated. Using the Chinese remainder theory with the unity difference between the two selected relative moduli, the scanning pattern can be physically implemented through a modification of any raster-scan display device. The mapped signal can then be optically processed in a standard optical image processor for its wavelet or Gabor processing. Advantages and shortcomings were analyzed.

3.5 Applications to Submicron Lithography

Analytic and algorithmic methodologies were developed for submicron filtering, image reconstruction, and image transfer in vector and scalar forms for study as to their applicability to micron lithography. This study contained two parts. In the first a projected signal had to be addressed in a scalar as well as a vector form and a Gabor representation of the final line profile had to be computed. The second task included a representation of electron beam applied to submicron masks, such that optimal intensity evaluation at each pixel will allow analysis of improved resolution.

We will describe in detail the several stages of the study.

In first part, we developed and formulated the algorithm or the aerial image methodology in a scalar form for partially coherent light with .5 micron line width. The partial coherence was addressed through the "mutual coherence function" that correlates the various light source elements. After performing a stationary phase calculation, we obtained this function in terms of Bessel function of order 1. Each entry in the four-fold integral final result is a two-fold integral. Thus, a fast algorithm was needed to avoid extremely expensive computations. Variable Gauss-Legendre quadrature with accuracy control coupled with the interior integrations order was chosen and implemented.

In the next stage, the aerial image for vectorial electric field was investigated and implemented. The rays were traced through the entrance as well as the exit pupil. The system is more complicated due to two extra integrations. We decided to lower the order of the innermost loops and check their accuracy via an averaged monte-carlo simulation of the integral. The polarization partial coherence matrix was formulated and the necessary algorithm developed and tested.

In the next stage, attention was given to the exposure systems of electron beams. We expressed the aerial image of E-beam as a linear combination of double Gaussian in each pixel due to back-scattering. The aerial image is thus expressed as a Gabor expansion and is ready made for data compression algorithm. The Haar basis and its corresponding wavelet basis failed to achieve optimal response. A projection system has been implemented, using smooth "hat" functions basis with correct polarity, thus allowing optimal resolution and data compression. A new data compression algorithm was developed.

The aerial image algorithms were generalized to include models with time-dependent absorption. The mechanism employed is a time dependent filter. In other words, the filter is being degraded gradually within its lifetime. The kinetics of the filter is obtained via fifth order Adams-Bashforth algorithm, and the field is refiltered at each time step. The necessary algorithm has been developed and implemented.

A vector Maxwell equation solver has been developed using spectral elements methods. An image is constructed and is deblumed through the dynamic filter developed earlier. The moving format is expressed as an eikonal system with a curvature controlled velocity of propagation, and this formalism lends itself to self improvement via a WKB scheme with the leading term obtained from the eikonal system. The corresponding PDEs have been analyzed.

The previous results have been integrated into a system of nonlinear reaction-diffusion equations for the post-exposure baking and a second system of dissolution. The reaction part has been solved analytically, and all the filtered concentrations are expressed explicitly in terms of the first concentration, which in turn is expressed implicitly. A very fast LV-decomposition algorithm has been developed to address finite-size effects while diffusion is deconvolved using our earlier bases.

4 Summary of Results

- Code in Fortran and C implementing adaptive selection of time-frequency shifted Gaussian and Hermite functions as analyzing signals relative to an incoming signal.

- Code testing the reliability of the adaptive selection procedure with comparisons to direct standard methods.

- Code computing generalized biorthogonals and testing the numerical stability and robustness of Gabor coefficient computation.

- An acousto-optic processor to implement Gabor coefficient computation of one-dimensional short signal sequences. The single-sided exponential was selected as the Gabor window for the detection of transients.

- An optical processor was built and tested to compute the wavelet expansion and its inverse.

- detail comparison of Gabor and wavelet transforms was undertaken in term of their complexity of optical implementations.

- Methods proposed and analyzed for optical processing of Gabor and wavelet transforms of one-dimensional data sequences using the Chinese remainder theorem.

- Analytic and algorithmic methodologies were developed for submicron filtering, image reconstruction, an image transfer and were applied to submicron lithography.

5 Announced Results

The participants in this project have published their results in many papers both in refereed journals and in invited articles in proceedings. We have listed most significant below.

5.1 Digital Computations

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5.2 Optical Results

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6. Zhang, Y. and Li, Y., *Coherent optical processing of Gabor and Wavelet expansions of 1D and 2D signals*. Optical Engineering, **31**, 1992, 1865-1885.

7. Zhang, Y. Li, Y. and Caulfield, H. J. *Optical implementation of long data stream convolution*, submitted to Applied Optics for publication.

Appendix

A Zak transform

For simplicity, we will describe several basic properties of the Zak transform of one-dimensional signals. Multidimensional extensions follow by the same arguments [21, 31, 34, 37, 38].

The *Zak transform* of a signal $f \in L^2(\Re)$ is the function of two variables $Z(f)(x,y)$, $x, y \in \Re$, defined by the formula

$$Z(f)(x,y) = \sum_k f(y+k)e^{2\pi ikx}.$$

$Z(f)(x,y)$ can be interpreted as the discrete Fourier transform at x of the sequence $f(y+k)$, $k \in \mathbf{Z}$ and hence contains joint time-frequency information about the signal f. The functional equations

$$Z(f) = (x+1,y) = Z(f)(x,y)$$

$$Z(f) = (x,y+1) = e^{-2\pi ix}Z(f)(x,y)$$

imply that the Zak transform is completely determined by its values on the unit square. We define the inner product of two Zak transforms by the formula

$$< Z(f_1), Z(f_2) >= \int_0^1 \int_0^1 Z(f_1)(x,y)Z^*(f_2)(x,y)dxdy.$$

Theorem 1 *For f_1 and f_2 in $L^2(\Re)$,*

$$< f_1, f_2 >=< Z(f_1), Z(f_2) >$$

Proof:

$$\begin{aligned}
< Z(f_1), Z(f_2) > &= \sum_j \sum_k \int_0^1 f_1(y+j)f_2^*(y+k)dy \int_0^1 e^{2\pi i(j-k)x}dx \\
&= \sum_j \int_0^1 f_1(y+j)f_2^*(y+k)dy \\
&= \int_{-\infty}^{\infty} f_1(y)f_2^*(y)dy \\
&= < f_1, f_2 > .
\end{aligned}$$

The following theorem characterizes the space of Zak transforms.

Theorem 2 *If*

$$\int_0^1 \int_0^1 \mid F(x,y) \mid^2 dx dy < \infty.$$
$$F(x+1,y) = F(x,y)$$
$$F(x,y+1) = e^{-2\pi i x} F(x,y)$$

then F is the Zak transform of a unique function $f \in L^2(\Re)$.

Proof: By the functional equations, we have

$$F(x,y) = \sum_r f_r(y) e^{2\pi i r x},$$

where

$$f_r(y+1) = f_{r+1}(y).$$

The Zak transform and the Fourier transform are related by the following formula.

Theorem 3

$$Z(\hat{f})(x,y) = e^{2\pi i x y} Z(f)(-y,x)$$

Proof: Apply the proceeding formula to

$$G(x,y) = e^{-2\pi i r x y} Z(f)(-y,x).$$

Define

$$f_{mn}(t) = f(t-n) e^{2\pi i m t}, \quad m, n \in \mathbf{Z}.$$

Theorem 4

$$Z(f_{mn})(x,y) = Z(f)(x,y) e^{2\pi i (nx+my)}$$

Proof:

$$\begin{aligned} \sum_k f_{mn}(y+k) e^{2\pi i k x} &= \sum_k f(y+k-n) e^{2\pi i m y} e^{2\pi i k x} \\ &= (\sum_k f(y+k) e^{2\pi i k}) e^{2\pi i (nx+my)}. \end{aligned}$$

The application of the Zak transform to time-frequency representation is the consequence of two fundamental formulas which we state without proof.

1st Fundamental Formula

$$Z(f)(x,y) Z^*(g)(x,y) = \sum_{m \in \mathbf{Z}} \sum_{n \in \mathbf{Z}} < f, g_{mn} > e^{2\pi i (nx+my)}$$

Applications

- $Z(f)(x,y)$ vanishes only on a set of measure 0 implies the set

$$\{g_{mn}: \quad m,n \in Z\}$$

spans $L^2(\Re)$.

- $| Z(g)(x,y) | \equiv 1$, *i.e.*, iff the set $\{g_{mn}: \quad m,n \in Z\}$ is an orthonormal basis of $L^2(\Re)$.

The second fundamental formula is directly tied to Gabor expansions, *i. e.*, expansions of the form

$$f = \sum_{m\in Z}\sum_{n\in Z} b_{mn} g_{mn}. \tag{1}$$

We will assume that the coefficients in (1) satisfy

$$\sum_{m\in Z}\sum_{n\in Z} | b_{mn} |^2 < \infty. \tag{2}$$

In this case, the series in (1) converges to f in $L^2(\Re)$.

Applying the Zak transform to both sides of (1) we have

2nd Fundamental Formula

$$Z(f)(x,y) = Z(g)(x,y) \sum_{m\in Z}\sum_{n\in Z} b_{mn} e^{2\pi i(nx+my)}.$$

The series converges in $L^2(I^2)$ to a doubly periodic norm-square summable function over the unit square. Multiplying both sides by $Z^*(g)(x,y)$, we have the following.

$$\sum_m \sum_n < f, g_{mn} > e^{2\pi i(nx+my)} = | Z(g)(x,y) |^2 \sum_m \sum_n a_{mn} e^{2\pi i(nx+my)}. \tag{3}$$

The right hand side can be viewed as a convolution.

For a fixed analyzing signal g, a signal $f \in L^2(\Re)$ need not have an expansion satisfying (2). The second fundamental formula is the main tool for deriving criteria on f and g guaranteeing such an expansion and for designing algorithms computing the Gabor coefficients b_{mn}, $m,n \in Z$. In fact, if the quotient

$$\frac{Z(f)}{Z(g)} \tag{4}$$

is norm-square summable over the unit square then the Fourier coefficients

$$b_{mn} \equiv \int_I \int_I \frac{Z(f)(x,y)}{Z(g)(x,y)} e^{-2\pi i(nx+my)} dx dy \tag{5}$$

satisfy (1) and (2). The problem is that if $Z(g)$ is continuous then it must vanish at some point, a result proved in [5] and henceforth called the Zero theorem. The quotient (4) is not necessarily norm-square summable.

The Zero theorem also lies at the heart of the synthesis problem of a signal from its short-time Fourier transform. The inner products, $< f, g_{mn} >$, m, $n \in \mathbf{Z}$, determine the product $Z(f)Z^*(g)$ *a. e.* but division by $Z(g)$ to recover $Z(f)$ is not necessarily numerically stable. For example, uniformly sampling both sides of the first fundamental formula over the unit square can produce samples of the product in terms of the 2-dimensional finite Fourier transform of periodized inner products but as we increase the resolution, zeros of $Z(g)$ will be approached whenever $Z(g)$ is continuous. Nonuniform sampling to avoid these zeros results in numerically unstable computation of product samples.

For fixed $g \in L^2(\Re)$, with $Z(g)$ continuous, there exists $f \in L^2(\Re)$ which do not admit Gabor expansions of the form (1) and which cannot be resynthesized from short-time Fourier transform. The special case of $g = e^{-\pi t^2}$ was considered in detail in [5, 30], where it was found that every $f \in L^2(\Re)$ with $Z(f)$ continuous has a Gabor expansion if we allow half integer shifts in the time or frequency variable, *i.e.*, n ranges over $\frac{1}{2}Z$. The main results in [30] give a precise count of the number of half integer shifts required to maintain various degrees of smoothness in the Gabor expansion. In [18], along with many other results, an account of similar ideas were presented in the language of frames.

In the following subsections, we will review more deeply these ideas and extend them to provide tools for an adaptive Gabor theory and for the design of efficient Gabor expansion implementation.

A.1 Analyzing Signal Parameters

Two important analyzing signal parameters are

- the zero set of their Zak transform
- the deviation of the absolute value of their Zak transforms from unity.

The zero set affects the existence and compactness of Gabor expansions while the deviation is a measure of the defect from orthonormality of the wavelet system.

The results in [5] provide a parameterized family of analyzing signals characterized by having Zak transforms with unique 'analytic' zeros in the unit square. These results can be extended to define larger families of Gaussian-based analyzing signals having more intricate and pre-assigned zero sets and indicate general methods for constructing non-Gaussian based analyzing signals.

The Zak transform of the Gaussian $g(t) = e^{-\pi t^2}$ can be written as

$$Z(g)(x, y) = e^{-\pi y^2} \nu(x, y)$$

where ν is the classical theta function of characteristic $(0,0)$

$$\nu(z) = \sum_{r \in Z} e^{-\pi r^2} e^{2\pi i r z}, \quad z = x + iy.$$

It is well-known that ν is an entire function having a unique zero in the unit square at $x = y = \frac{1}{2}$.

In general, if $g_{uv}(t) = g(t-v)e^{2\pi i u t}$, $u, v \in \Re$, then

$$Z(g_{uv})(x,y) = e^{2\pi i u y} Z(g)(x+u, y-v). \tag{6}$$

Applying (6) to the Gaussian $g(t) = e^{-\pi t^2}$, we see that the collection

$$\{g_{uv} : \quad 0 \leq u, v < 1\}$$

of time-frequency translates of the Gaussian have Zak transforms with unique 'analytic zeroes' in the unit square and each point in the unit square is the Zak transform zero of exactly one function in the collection.

Gaussian-based signals having more complicated zero sets can be constructed using the following formula

$$Z(g_{mn})(x,y) = e^{2\pi i(nx+my)} Z(g)(x,y), \quad m, n \in Z. \tag{7}$$

If $P(x,y)$ is the trigonometric polynomial

$$P(x,y) = \sum_m \sum_n a_{mn} e^{2\pi i(nx+my)},$$

where the double summation is finite, then (7) implies

$$P(x,y)Z(g)(x,y) = Z(\sum_m \sum_n a_{mn} g_{mn})(x,y). \tag{8}$$

The zero set of the Zak transform of

$$h = \sum_m \sum_n a_{mn} g_{mn}$$

is the union of the zero set of $Z(g)(x,y)$ and the zero set of the trigonometric polynomial $P(x,y)$.

This result can be used to construct signals having preassigned Zak transform zero sets. For fixed g with $Z(g)$ continuous, the zero set of $Z(g)$ is also included. Applied to the Gaussian case, $g(t) = e^{-\pi t^2}$, we can build an extensive collection of signals with a wide variety of Zak transform zero sets.

By creating a families of signals having a wide range of Zak transform zero sets, we introduce into the selection procedure a criteria based on matching signal to analyzing signal Zak transform zero sets. One measure of the effectiveness of this approach will be the compactness of the representation and the efficiency of the computation.

Example 1 The trigonometric polynomial

$$P_1(x,y) = 1 + ie^{2\pi ix} - e^{2\pi i(x-y)} - ie^{-2\pi iy}$$

has exactly two zeros in the unit square, at $(0,0)$ and $(\frac{1}{2},\frac{1}{2})$. The product $P_1(x,y)Z(g)(x,y)$, $g = e^{-\pi t^2}$, has a second order zero at $(\frac{1}{2},\frac{1}{2})$ and a first order zero at $(0,0)$. Translating g gives rise to a product which has three first order zeros.

Example 2 The trigonometric polynomial

$$P_2(x,y) = 2 + ie^{2\pi ix} + ie^{2\pi iy}$$

has exactly one zero in the unit square, at $(\frac{1}{4},\frac{1}{4})$. The product $P_2(x,y)Z(g)(x,y)$, $g = e^{-\pi t^2}$, has first order zeros at $(\frac{1}{4},\frac{1}{4})$ and $(\frac{1}{2},\frac{1}{2})$.

The general results (6), (7) and (8) can be applied to signals other than the Gaussian. In preliminary studies, we have carried out these constructions on such 'naturally' defined signals as one-sided exponentials, Hermite functions and finite interval restrictions of periodic signals. We intend to increase the list of naturally defined signals to other special functions and to include digital signals which have played important roles in applications.

In [5], partition of unity arguments matched to the underlying Heisenberg group structure were used to prove several important results. These ideas can also be used to construct signals whose Zak transforms have pre-assigned zero sets and pre-assigned Taylor expansions. The feasibility of constructing signals in this way is under study.

In general, the collection

$$\{g_{mn} : \quad m,n \in Z\}$$

is not orthogonal, as one can see from the Gaussian $g(t) = e^{-\pi t^2}$. The condition for orthonormality

$$\mid Z(g)(x,y) \mid \equiv 1, \quad a.e., \tag{9}$$

is a simple consequence of the first fundamental formula applied to $f = g$:

$$\mid Z(g)(x,y) \mid^2 = \sum_m \sum_n < g, g_{mn} > e^{2\pi i(nx+my)}.$$

By the zero theorem, if $Z(g)$ is continuous then (9) can not hold. A more exact statement about the continuity of $Z(g)$ expressed in terms of rate of decay of g and orthonormality is given by Balian's theorem. There are several ways of measuring defect from orthonormality. For example

$$\int_0^1 \int_0^1 \mid\mid Z(g)(x,y) \mid^2 -1 \mid^2 dxdy = \sum_m \sum_n{}' \mid< g, g_{mn} >\mid^2,$$

where $\sum_m \sum_n'$ denotes summation over all $m,n \in Z$ except $m = n = 0$, measures the energy in the norm-square sense of the inner product $< g, g_{mn} >$, $m,n \in Z$, except $m = n = 0$.

A.2 Selection Procedure

A signal f has a Gabor expansion

$$f = \sum_{m \in Z} \sum_{n \in Z} a_{mn} g_{mn} \tag{10}$$

satisfying

$$\sum_{m \in Z} \sum_{n \in Z} | a_{mn} |^2 < \infty$$

if and only if

$$\frac{Z(f)}{Z(g)} \in L^2(I^2). \tag{11}$$

The Gabor coefficients in (10) are the Fourier coefficients of (11). In general, the smoother the quotient, the more rapid the decay of the Gabor coefficients and the more 'compact' the Gabor expansion (10) in the sense that finite partial sums should better represent f. In [1], a computer experiment verified these results for the special case of the Gaussian as an analyzing signal. In fact, it was shown that if the quotient is not in L^2 then a 'good' Gabor expansion does not exist.

Selection Procedure For a given signal f, compute the Zak transform of f and determine its zero set. From a library, choose $g's$ whose Zak transform zero set is contained in the zero set of $Z(f)$. Consider the quotients (11). The smoother the quotient, the more rapid the decay of the Gabor coefficients in the sense described by harmonic analysis theorems.

Computer experiments have verified these comments in the special case of the Gaussian g with signals f taken from the Hermite functions. It should be pointed out that although the theory is about analog signals computation must take place on the discrete level.

If the Zak transform of f has two zeros in the unit square then the harmonic analysis criteria can point, say, to a Gaussian type g having exactly one of these zeroes in the unit square but from a more pattern recognition point of view it might better to choose a g whose Zak transform zeroes match exactly the zero set of $Z(f)$. Preliminary studies bear out this possibility. A Hermite function having exactly three zeroes in the unit square was taken as the the analyzing signal. Higher order Hermite functions were expanded relative to the Gaussian and relative to the analyzing Hermite function. The Hermite analyzing signal produced a more compact representation in most cases.

When available, Taylor series expansions at the zeroes can be matched as much as possible.

A.3 Generalized Gabor Expansions

A signal f will not always have a Gabor expansion relative to an analyzing signal g. The selection procedure of the previous section establishes rules for degrees of compatibility between f and g based on the comparison of their Zak transforms, either by smoothness of the quotient or optimal matching of zeroes. However, it can happen in some applications that a fixed g must be taken. The problem is then to modify the definition of Gabor expansions so that an f can be expanded as a modified Gabor expansion. Again our goal is to produce an adaptive procedure which depends on both f and g.

The main idea is that the definition given of Gabor expansion implicitly depends on the lattice Z and that by extending the lattice, we can guarantee Gabor expansions. More precise results for the Gaussian case appear on [30] which include algorithms of Gabor coefficient computation will be described below.

Set $g_1 = e^{-\pi t^2}$ and $g_2 = (g_1)_{0,\frac{1}{2}} = e^{-\pi(t-\frac{1}{2})^2}$. The Zak transforms of g_1 and g_2 vanish uniquely at the point $x = y = \frac{1}{2}$ and $x = \frac{1}{2}$, $y = 0$, respectively, in the unit square. Denote the r-times continuously differentiable functions in the plane by C^r. The first result we have is that if $Z(f) \in C^r$, then

$$Z(f) = pZ(g_1) + qZ(g_2),$$

where p and q are trigonometric series in C^r, *i.e.*,

$$p(x,y) = \sum_{m\in Z}\sum_{n\in Z} a_{mn}e^{2\pi i(nx+my)}, \tag{12}$$

$$q(x,y) = \sum_{m\in Z}\sum_{n\in Z} b_{mn}e^{2\pi i(nx+my)}.$$

If $r \leq 2$, say, then we can write

$$f = \sum_{m\in Z}\sum_{n\in Z} a_{mn}(g_1)_{mn} + \sum_{m\in Z}\sum_{n\in Z} b_{mn}(g_2)_{mn}, \tag{13}$$

where

$$\sum_m\sum_n \mid a_{mn} \mid^2 < \infty,$$

$$\sum_m\sum_n \mid b_{mn} \mid^2 < \infty.$$

Since $g_2 = (g_1)_{0,\frac{1}{2}}$, we can write (13) as

$$f = \sum_{m\in Z}\sum_{n\in Z} c_{mn}g_{m,\frac{n}{2}},$$

where $c_{m2n} = a_{mn}$ and $c_{m2n+1} = b_{mn}$. The lattice of the expansion is $Z \times \frac{1}{2}Z$. The overall effect is to double the sampling rate. Similar results can be found in the language of frames in [18].

The proof of (12) as given in [5] depends on a partition of unity argument adapted to the Heisenberg group and easily generalizes. In particular, the choice of the Gaussian g_1 and g_2 are arbitrary and can be replaced by any two distinct time-frequency translates of the Gaussian $g = e^{-\pi t^2}$. In general, results of the form (12) depend solely on the disjointness of the zero sets of g_1 and g_2 and can be extended to any number of analyzing signals. Results of the form (13) require that these signals be related by time-frequency translates.

The central result in [30] is that q can always be taken as a trigonometric polynomial whose coefficients are explicitly computable. Uniqueness and algorithms constructing p and q are given in [30]. A quantitative relationship is established between the desired smoothness of p and the degree of the trigonometric polynomial q. Computer experiments have been carried out for the Gaussian. Other important special functions whose role is more than an analyzing signal but is also part of the application have been studied. In particular, Hermite functions are intrinsically interesting in many applications including image coding, computer vision and human visual perception [22].

The main result in [30] is that for $Z(f) \in C^3$, we can uniquely write

$$Z(f) = pZ(g_1) + qZ(g_2),$$

where $p \in C^2$ and q is a trigonometric polynomial of degree 3 whose coefficients are explicitly computable in terms of the partial derivatives of $Z(f)$ at $x = y = \frac{1}{2}$. Recent results [20] have implemented the computation of p and q soley using Zak transform methods.

The Fourier transform plays a major part in [1] and [30]. Functions are decomposed into their eigen vector subspaces relative to the Fourier transform which on the plane is given by the linear operator

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

A.4 Digital Computations

Gabor expansions must be finitized for digital computations. Recent efforts [4, 23, 32] have subjected Gabor-expansions to the same periodization and sampling procedures which underlie the classical Fourier sampling theory and have contrasted results to standard truncation and sampling. The main digital Gabor expansion formula will be described below but the main point to be emphasized is that the analyzing signal translates undergo periodization creating overlap. In the Fourier theory, the only relevant periodization occurs in the expansion coefficients since the basis functions remain unchanged under periodization. The appearance of periodized analyzing signals in the digital Gabor expansion case is a reflection of the nonlinearity of such expansions.

Suppose a signal f has a Gabor expansion

$$f = \sum_{m \in Z} \sum_{n \in Z} a_{mn} g_{mn}$$

satisfying $\sum_{m \in Z} \sum_{n \in Z} \mid a_{mn} \mid^2 < \infty$. The Gabor coefficients are uniquely determined by the norm-square finite energy condition. In general, the system

$$\{g_{mn} : \quad m, n \in Z\}$$

is not orthogonal so straightforward computation of Gabor coefficients is not always possible. The design of accurate and stable algorithms for computing Gabor coefficients (analysis) and for computing input signal from Gabor transform methods useful signal processing tools. This step is intimately tied to the form and meaning assigned to digital Gabor expansions. We will review briefly how periodization sampling procedures force such digital Gabor expansions. Such expansions as contrast to more standard forms highlight the tradeoff between sampling rate and aliasing errors. For simplicity, choose integers $M > 0$ and $N > 0$ as sampling rate and periodization interval. Periodize f mod N.

$$f_N(t) \equiv \sum_{j \in Z} f(t + jN),$$

and sample f_N at rate $\frac{1}{M}$.

$$f_N(\frac{m}{M} + k) = \sum_{j \in Z} f(\frac{m}{M} + k + jN), \tag{14}$$

$$0 \leq m < M, \quad 0 \leq k < N.$$

The resulting L-tuple of values, $L = MN$, is the digital signal corresponding to f. The independent parameters M and N are usually fixed by a priori information in a given application.

The samples (14) are related to Zak transform samples by

$$Z(f)(\frac{n}{N}, \frac{m}{M}) = \sum_{k=0}^{N-1} f(\frac{m}{M} + k) e^{2\pi i k \frac{n}{N}}.$$

By the second fundamental formula, these samples equal

$$Z(g)(\frac{n}{N}, \frac{m}{M}) \sum_{s=0}^{N-1} \sum_{r=0}^{M-1} A_{rs} e^{2\pi i (s \frac{n}{N} + r \frac{m}{M})} \tag{15}$$

where

$$A_{rs} = \sum_{r' \in Z} \sum_{s' \in Z} a_{r+r'M, s+s'N}.$$

A similar argument is the basis for finitizing the Fourier transform. However, since

$$Z(g_{rs})(\frac{n}{N}, \frac{m}{M}) = Z(g)(\frac{n}{N}, \frac{m}{M})e^{2\pi i(s\frac{n}{N}+r\frac{m}{M})}$$

we can rewrite (15) as

$$\sum_{s=0}^{N-1}\sum_{r=0}^{M-1} A_{rs}Z(g_{rs})(\frac{n}{N}, \frac{m}{M}).$$

By formula (14), we have the desired *finite Gabor expansion*

$$f_N(\frac{m}{M}+k) = \sum_{s=0}^{N-1}\sum_{r=0}^{M-1} A_{rs}(g_{rs})_N(\frac{m}{M}+k), \tag{16}$$

$$0 \le m < M,\ 0 \le n < N.$$

The periodized analyzing signal translates

$$(g_{rs})_N(\frac{m}{M}+k),\quad 0 \le r < M,\ 0 \le s < N$$

form a basis for signal expansion in $L(\mathbf{Z}/L)$, $L = MN$, and were introduced in [4]. Algorithms for computing the Gabor coefficients of the finite Gabor expansion (16) were derived in the work and increased resolution procedures were established.

The overlapping of the basis signals $(g_{rs})_N$ around the sampling interval introduces additional programming effort and at times aliasing errors which must however be accounted for. Code provided in this proposal will compare the benefits of this additional effort to standard truncation-sampling approaches and make the optimal decision.

A.5 New Algorithms for Computing Gabor Coefficients

Several algorithms are by now standard for computing Gabor coefficients. A summary of two appear in [3] along with the basic constraints and tradeoffs.

The *biorthogonal* approach introduced in [12] depends on solving the equation

$$Z(h)(x,y)Z^*(g)(x,y) = 1. \tag{17}$$

The function h is called a biorthogonal of g. If $Z(g)$ is continuous then a solution $h \in L^2(\Re)$ need not exist. Neglecting this point for the moment, if some h satisfying (17) is found then the Gabor coefficients can be computed by the formula

$$a_{mn} = < f, h_{mn} >, \tag{18}$$

which is an immediate result of the first fundamental formula.

The problem of this approach for continuous $Z(g)$ is that since h need not be in $L^2(\Re)$ the computation (18) can be difficult to carry out. We propose a generalization which has the effect of localizing Gabor coefficient computation and producing stable local computations by 'delaying' the instability to regions removed from some initialization. We will explore the approach for the Gaussian $g = e^{-\pi t^2}$.

Although (17) cannot be solved for $h \in L^2(\Re)$, we can solve

$$Z(h_1)(x,y)Z^*(g)(x,y) = 1 + e^{2\pi i x}, \tag{19}$$

$$Z(h_2)(x,y)Z^*(g)(x,y) = 1 + e^{2\pi i y}, \tag{20}$$

for h_1 and h_2 in $L^2(\Re)$. In this case, $h_2 = \hat{h}_1$, the Fourier transform of h_1 but this will only be the case when $g = \hat{g}$. From the first fundamental formula

$$< h_1, g_{mn} >= \begin{cases} 1, & m = n = 0 \ \ or\ m = 0, n = 1 \\ 0, & otherwise, \end{cases}$$

$$< h_2, g_{mn} >= \begin{cases} 1, & m = n = 0 \ \ or\ m = 1, n = 0 \\ 0, & otherwise, \end{cases}$$

The algorithm proceeds by first precomputing h_1 and h_2 and then computing

$$b_{rs} =< f, (h_1)_{rs} >, \quad r, s \in Z, \tag{21}$$

$$c_{r0} =< f, (h_1)_{r0} >, \quad r \in Z. \tag{22}$$

Formula (19) and (20) imply

$$b_{rs} = a_{rs} + a_{r,s+1}, \tag{23}$$

$$c_{rs} = a_{rs} + a_{r+1,s}. \tag{24}$$

Assuming that a_{00} is known, the computations (21) and (22) can be placed (23) in and (24) to compute all a_{rs}.

The computations are locally stable about a_{00} but by analyzing the impulse response related to the difference equations we see that the resulting transfer function is only marginally stable. The choice of initialization at a_{00} is arbitrary and similar results can be derived about any initialization. Stable global algorithms require a precise understanding of the branching that can occur at points away from an initialization and rules for providing a new initialization at the 'boundaries of stability'.

An increasing degree of smoothness of h can be achieved by replacing $1 + e^{2\pi i x}$ in (20) by trigonometric polynomials having higher order zeroes at $x = y = 1/2$. However, the increasing degree of smoothness of h is paid for by an increasing complexity of the synthesis algorithm including increasing initializations and instability as measured by the impulse response poles on the unit circle (equal to the degree of the poles). The quantification of the tradoff between increased smoothness and complexity / instability of synthesis algorithm including measurement of bounds for stability will be studied in this proposal.

B The Ambiguity Function

The results of this section were first proved in a series of papers [26, 27, 28, 29] in the early part of the 1960s and provided much of the theoretic framework during the following decade for radar signal synthesis. Throughout, we work with the unsymmetric form of the cross-ambiguity function of two signals $f,\ g \in L^2(\Re)$.

$$A(f,g)(u,v) = \int f(t)g^*(t-v)e^{2\pi iut}dt. \tag{25}$$

There are two important ways to view cross-ambiguity functions. First we can construct the parametrized product

$$h_v(t) = f(t)g^*(t-v) \tag{26}$$

and form $A(f,g)(u,v)$ as the Fourier transform of $h_v(t)$:

$$A(f,g)(u,v) = \int h_v(t)e^{-2\pi iut}dt. \tag{27}$$

In signal processing, $A(f,g)(u,v)$ is called the short-time Fourier transform. The function $g(t)$ is viewed as a window at the origin and $g(t-v)$ as a sliding window which for a particular v is centered at v. The parametrized product $h_v(t)$ presents a windowed version of f around v.

It is easy to show that $A(f,g)(u,v) \in L^2(\Re)$ and

$$\| A(f,g) \|_2^2 = \| f \|^2 \| g \|^2 . \tag{28}$$

In fact

$$< A(f_1,g_1), A(f_2,g_2) > = < f_1, f_2 >< g_2, g_1 > . \tag{29}$$

Also, we can recover the windowed function by the Fourier inversion formula

$$f(t)g^*(t-v) = \int A(f,g)(u,v)e^{2\pi itu}du. \tag{30}$$

We can go one step further by taking the Fourier transform on both sides with respect to the v variable :

$$f(t)(\hat{g})^*(y)e^{2\pi iyt} = \int\int A(f,g)(u,v)e^{2\pi i(tu-yv)}dudv. \tag{31}$$

This *separation* property under the Fourier transform has been shown to be necessary and sufficient for a cross-ambiguity function. The left-hand side of (31) has also been identified as a complex energy density function for complex envelope f and g. That is

$$f(t)(\hat{g})^*(y)e^{2\pi iyt}\delta y\delta t \tag{32}$$

can be interpreted as the differential complex energy in a bin δy by δt located at y and t, where the total complex energy is simply the inner product

$$E_c =< f, g >= \int f(t)g^*(t)dt. \tag{33}$$

Equation (31) shows that this measure of energy density may be computed by evaluating the two-dimensional Fourier transform of the cross-ambiguity function of f and g at the point (t, y). In this light it should not be surprising to find $A(f, g)$ appear in other ways that relate to signal energy – see the discussion of Weyl-Heisenberg frames in section 4 of this paper.

A second way of writing the cross-ambiguity function is as an inner product

$$A(f, g)(u, v) =< f, g_{uv} >, \tag{34}$$

where $g_{uv}(t) = g(t-u)e^{2\pi iut}$. Since

$$\widehat{g_{uv}}(t) = e^{2\pi iuv}(\hat{g})_{-uv}, \tag{35}$$

we have by the Plancheral theorem that

$$\begin{aligned} A(f, g)(u, v) &= < \hat{f}, \widehat{g_{uv}} > \\ &= e^{-2\pi iuv} \int \hat{f}(t)\hat{g}(t-v)e^{2\pi ivt}dt \\ &= e^{-2\pi iuv} A(\hat{f}, \hat{g})(-v, u). \end{aligned}$$

If we restrict (u, v) to a lattice of points (mM, nN), $m, n \in \mathbf{Z}$, the corresponding set of samples

$$A(f, g)(mM, nN) =< f, g_{mM,nN} >, \quad m, n \in \mathbf{Z} \tag{36}$$

is called the discrete short-time Fourier transform of f relative to the analyzing signal g and the lattice determined by (M, N). The set of functions

$$\{g_{mM,nN} : m, n \in \mathbf{Z}\}, \tag{37}$$

is usually called a Weyl-Heisenberg wavelet system but in this work we will call it simply a Gabor wavelet system. It will be denoted by (g, M, N). We can ask whether from the set of inner products (36) a numerically stable algorithm reconstructs or approximates f. As explained in [17] the question can be answered affirmatively if and only if the following condition is satisfied :

Frame Condition There exist constants $0 < A \leq B < \infty$ such that for all $f \in L^2(\Re)$,

$$A \parallel f \parallel^2 \leq \sum_m \sum_n |< f, g_{mM,nN} >|^2 \leq B \parallel f \parallel^2 . \tag{38}$$

In this case, we call (g, M, N) a frame and A and B frame bounds. If, for some constant $0 < B < \infty$,

$$\sum_m \sum_n |< f, g_{mM,nN} >|^2 = B \parallel f \parallel^2 \tag{39}$$

whenever $f \in L^2(\Re)$, we say that (g, M, N) is a tight frame. Condition (38) easily translates into a statement about cross-ambiguity function samples :

$$A \parallel f \parallel^2 \leq \sum_m \sum_n | A(f,g)(mM, nN) |^2 \leq B \parallel f \parallel^2 . \tag{40}$$

Our tools for analyzing frame conditions are based on the following analog summarized in [16].

Radar Theorem (1960)

Theorem 5 *If f, $g \in L^2(\Re)$, then*

$$\int\int | A(f,g)(u,v) |^2 e^{-2\pi i(xu+yv)} dudv \tag{41}$$

$$= A(f)(y,-x)A^*(g)(y,-x).$$

Theorem 6 *If f_1, f_2, g_1, $g_2 \in L^2(\Re)$, then*

$$\int\int A(f_1,f_2)(u,v)A^*(g_1,g_2)(u,v)e^{-2\pi i(xu+yv)} dudv \tag{42}$$

$$= A(f_1,g_1)(y,-x)A^*(f_2,g_2)(y,-x).$$

In [24], the Poisson summation formula was used to derive discrete analog of theorems 1 and 2. We describe these discrete analogs. The exact conditions on f and g required for these formulas are given in [24].

Theorem 7 $$\sum_n f(t+nN)g^*(t+nN-\frac{m}{M})$$ (

$$= \frac{1}{N}\sum_n A(f,g)(\frac{n}{N},\frac{m}{M})e^{2\pi i \frac{n}{N}t}, \quad m \in M.$$

Theorem 8 $$\sum_m \sum_n | A(f,g)(mM,nN) |^2$$ (

$$= \frac{1}{MN}\sum_m \sum_n A(f)(\frac{n}{N},\frac{m}{M})A^*(g)(\frac{n}{N},\frac{m}{M}).$$

Theorem 9 $$\sum_m \sum_n A(f_1,g_1)(mM,nN)A^*(f_2,g_2)(mM,nN)$$ (

$$= \frac{1}{MN}\sum_m \sum_n A(f_1,f_2)(\frac{n}{N},\frac{m}{M})A^*(g_1,g_2)(\frac{n}{N},\frac{m}{M}).$$

C Applications

We say $(g, \frac{1}{N}, 1M)$ satisfies **Condition A**, if $A(g)(\frac{n}{N}, mM) = 0$, unless $m = n = 0$,

First we show that there exists $g \in L^2(\Re)$ satisfying condition A. Define

$$g_0(t) = \begin{cases} 1, & 0 \le t < N, \\ 0, & \text{otherwise.} \end{cases} \tag{46}$$

Since $N \le \frac{1}{M}$,

$$g_0(t)g_0^*(t - \frac{m}{M}) = 0, \quad \text{unless } m = 0. \tag{47}$$

Condition A follows from

$$\begin{aligned} A(g_0)(\frac{n}{N}, 0) &= \int_0^N e^{-2\pi i \frac{n}{N} t} dt \\ &= \begin{cases} N, & n = 0, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

In [24] the following result was proved.

Theorem 10 (g, M, N) *is a tight frame if and only if g satisfies condition A. In this the frame bound is*

$$B = \frac{\| g \|^2}{MN}. \tag{48}$$

The proof of the theorem and other results found in [24] rest on the application of the discrete formulas to control the middle sum of the frame condition by imposing constraints on the values of the auto-ambiguity of the analyzing signal g over the dual lattice. Usually, no condition except for square-integrability is imposed on f. The application of these ideas to the design of adaptive methods depends on imposing joint conditions on the auto-ambiguities of f and g on the dual lattice.

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